Chapter 4.1 part 3

F is a field   We study $F[x]$

In both rings $\mathbb{Z}$ and $F[x]$, the Fundamental Theorem of Arithmetic holds.

Th1.8 Every integer - element $\mathbb{Z}$ - except 0 and ± 1 can be written as a product of primes in an essentially unique way

Th 4.4 Every polynomial - element of $F[x]$ - except constants can be written as a product of irreducibles in an essentially unique way

Exceptions are the same: these are 0 and the units in the ring ($\mathbb{Z}$ or $F[x]$)

Euclid's Lemma (in $\mathbb{Z}$)

Th1.1 $a, b \in \mathbb{Z}$, $b > 0$

There exist unique $q, r \in \mathbb{Z}$ such that

$$a = bq + r \qquad 0 \leq r < b$$

either $r = 0$ or $0 < r < b$

Euclid's Lemma (in $F[x]$)

Th 4.6 $f, g \in F[x]$ $g \neq 0_F = 0_{F[x]}$

There exis unique $q, r \in F[x]$ such that

$$f = gq + r \qquad \text{either } r = 0_F \text{ or } \deg r < \deg g$$

Pf Uniqueness $f = g q_1 + r_1$

$f = g q_2 + r_2$ either $r_i = 0_F$ or $\deg r_i < \deg$ $i = 1, 2$

We'll use Ex 12 p 94 $\deg(r_1 \pm r_2) \le \max(\deg r_1, \deg r_2)$

$$g(q_1 - q_2) = r_2 - r_1$$

Assume, for the sake of a contradiction, that $q_1 \neq q_2$, $r_1 \neq r_2$.

By Th 4.2,

$$\deg g + \deg(q_1 - q_2) = \deg(r_2 - r_1)$$

$\deg(r_2 - r_1) = \max(\deg r_1, \deg r_2) < \deg g$

$\deg(q_1 - q_2) \ge 0$ by the definition of degree.

Thus the equality cannot hold.

Wanted: $\underline{r_1 = r_2}$, $\underline{q_1 = q_2}$

If $r_1 = r_2$, then $g(q_1 - q_2) = 0_F$ implies $q_1 = q_2$ because $F[x]$ is an integral domain by Cor 4.3

If $q_1 = q_2$, then

$$r_2 - r_1 = g \cdot 0_F = 0_F$$